"Digital" entropy as an invariant under a refinement of the partition of the interval

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Let $I = [0, n]$ be a closed interval and let $T = \{0, 1, \ldots, n\}$ be a set of partition points of I. Let $f : I \to I$ be a piecewise-continuous Markov map with respect to the partition points T, that is, f satisfies that (1) f is strictly monotonic and continuous on each subinterval $I_i = (i, i+1)$ $i = 0, 1, \ldots, n-1$ and (2) both the right and left limits $f(i^+) = \lim_{x \to i^+} f(x)$ and $f(i^-) = \lim_{x \to i^-} f(x)$ are elements of T. Let A be a $0-1$ matrix of size $n \times n$ which is induced by f with T. Let $S = \{0 = s_0 < s_1 < \cdots < s_{m+n} = n\}$ be a partition of the interval $[0, n]$ such that S is a refinement of T (that is, $T \subset S$) and f with S also induces a $0-1$ matrx B of size $(m+n) \times (m+n)$.

Then we have the following results:

Proposition 1 Suppose that for any $s_i \in S - T$ there exists $k \in \mathbb{N}$ such that $f^k(s_i) \in T$, that is, $S - T$ consists of m eventually cyclic points. Then the characteristic polynomial $Ch_B(\lambda)$ of B is expressed as the product of the characteristic polynomial $Ch_A(\lambda)$ of A and λ^m, that is,

$$Ch_B(\lambda) = Ch_A(\lambda)\, \lambda^m.$$

Proposition 2 Suppose that $S - T$ consists of the orbit of an m-cycle induced by $(-)$ -signed closed walk in G_f. Then

$$Ch_B(\lambda) = Ch_A(\lambda)\, (\lambda^m + 1).$$

Proposition 3 Suppose that $S - T$ consists of the orbit of an m-cycle induced by $(+)$ -signed closed walk in G_f. Then

$$Ch_B(\lambda) = Ch_A(\lambda)\, (\lambda^m - 1).$$

Combining Propositions 1-3, we may have the following:

Proposition 4 Suppose that $S-T$ consists of m_0 of eventually cyclic points, m_i of $(-)$ -signed l_i-cycles $(i=1,\ldots,j)$, and m_i of $(+)$ -signed l_i-cycles $(i=j+1,\ldots,k)$, where $m_0+\sum_{i=1}^{k} m_i l_i = m$. Then

$$Ch_B(\lambda) = Ch_A(\lambda)\,\lambda^{m_0}\prod_{i=1}^{j}\left(\lambda^{l_i}+1\right)^{m_i}\prod_{i=j+1}^{k}\left(\lambda^{l_i}-1\right)^{m_i}.$$

Note: Proposition 1 implies that the "digital" entropy of θ does **not** depend on how to choose a refinement of the partition of the interval $[0, n-1]$, if the refined partition does not include any periodic orbit, except T, the orbit of the n-cycle with type θ.

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